UNIT - II

The conjugate of an Operator

Definition:- Let 'N' be a normed linear space and 'T' is the operator on 'N' ie, T is cont., linear transformation from N into itself and let N^* be the conjugate space of N then T induces a corresponding operator denoted by T^* on N^* and that operator is called "conjugate of T" if $T^*(f(x)) = f(T(x))$

Theorem:- Uniform Boundedness theorem (or) Banach - Steinhaus theorem

Statement:- Let 'B' be a banach space and 'N' is a normed linear Space. If $\{T_i\}$ is a non-empty set of cont., linear transformation of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N. For each vector 'x' in B then $\{\|T_i\|\}$ is a bounded set of numbers i.e, $\{T_i\}$ is bounded as a subset of $\mathcal{B}(B, N)$

Proof:- For each +ve integer 'n'

Let us define $F_n = \{x \mid x \in B, \|T_i(x)\| \le n \ \forall i\}$

Then F_n is a closed subset of B.

For, if $x \in F_n \Leftrightarrow \|T_i(x)\| \le n \ \forall i$ $\quad$ F_n is closed, $\|x\| \le r$ (or) mapping $x \in S(r)$

$\Leftrightarrow \|T_i(x)\| \in S_n(0)$ where $S_n(0)$ is closed sphere in N

$\Leftrightarrow x \in T_i^{-1}(S_n(0)) \ \forall i$

$\Leftrightarrow x \in \cap(T_i^{-1}(S_n(0))) \ \forall i$

$\therefore F_n = \cap(T_i^{-1}(S_n(0)))$

Since each T_i is cont., $\Rightarrow T_i^{-1}$ is also continuous. And $S_n(0)$ is closed in N

$\Rightarrow T_i^{-1}(S_n(0))$ is closed in B.

$\because$ Intersection of closed sets is closed.

$\Rightarrow \cap T_i^{-1}(S_n(0))$ is closed.

$\therefore F_n$ is closed.

Further, $B = \bigcup_{n=1}^{\infty} F_n$

for, if $B \ne \bigcup_{n=1}^{\infty} F_n$

$\Rightarrow \exists$ some $x \in B \ni x \notin \bigcup_{n=1}^{\infty} F_n$

$\Rightarrow x \notin F_n \ \forall n$

$\Rightarrow \|T_i(x)\| > n \ \forall i$

$\Rightarrow \{T_i(x)\}$ is not bounded subset of N.

which is a contradiction to the hypothesis

$B = \bigcup_{n=1}^{\infty} F_n$

Now by applying Baire's theorem, atleast one of the closure of a set F_n has non-empty interior.

ie, $\exists$ some integer $n_0 \ni \bar{F}_{n_0}$ has non-empty interior.

$\because F_{n_0}$ is closed we have $\bar{F}_{n_0} = F_{n_0}$

And hence F_{n_0} has non-empty interior.

Let $x_0 \in F_{n_0}$ is an interior point of F_{n_0}

$\because F_{n_0}$ is closed

$\Rightarrow \exists$ a closed sphere $S = \{x / \|x - x_0\| < r_0\} \subseteq F_{n_0}$ —(1), for every $x \in B$

Now if $\|y\| < 1$, then for any arbitrary fixed 'i'

$\|T_i(y)\| = \|T_i(\frac{z}{r_0})\|$, where $y = \frac{z}{r_0}$

$= \frac{1}{r_0} \|T_i(z)\|$

$= \frac{1}{r_0} \|T_i(z + x_0 - x_0)\|$

$\leq \frac{1}{r_0} \|T_i(z + x_0)\| + \|T_i(x_0)\|$

$\leq \frac{1}{r_0}(n_0 + n_0)$

$= \frac{1}{r_0}(2n_0)$

$\because \|T_i\| = \sup\{\|T_i(y)\| / \|y\| \leq 1\}$

$\leq \sup\{\frac{1}{r_0}(2n_0)\}$

$= \frac{1}{r_0}(2n_0)$

$\Rightarrow \{\|T_i\|\}$ is a bounded set of numbers.

Hence proved.

Theorem:B A non-empty subspace 'X' of a normed linear space is bounded $\Leftrightarrow f(x)$ is a bounded set of numbers, for each 'f' $\in N^*$

Proof:- Given that 'X' is a non-empty subset and 'N' is a normed linear space

claim:- 'X' is a bounded set of $N \Leftrightarrow f(x)$ is a bounded set of numbers for each $f \in N^*$

First suppose that 'X' is bounded subset of 'N'

By def, $\exists$ a +ve integer $K_1 \ni \|x\| \leq K_1, \forall x \in X$ — (1)

Now we have to prove that $f(x)$ is bounded set of numbers for each $f \in N^*$

$\because f \in N^* \Rightarrow$ 'f' is cont, linear transformation and hence bounded.

$\therefore$ 'f' is bounded.

$\Rightarrow \exists$ a +ve integer $K_2 \ni \|f(x)\| \leq K_2 \|x\|$

$\leq K_2 K_1$

$\therefore \|f(x)\| \leq K \; \forall x \in X$ $= K$

$\therefore f(x)$ is bounded set of numbers $\forall f \in N^*$

Conversely, suppose that $f(x)$ is bounded set of numbers $\forall f \in N^*$

Then we have to prove that $X = \{x_i\}$ is bounded set

Now we use natural embedding $x_i \rightarrow \{F_{x_i}(x_i)\}$ (as det of natural embedding)

to pass from 'x' to the corresponding subset $\{F_{x_i}\}$ of N^{**}

Since our assumption that $f(x) = \{f_x(x_i)\}$ is bounded for each 'f' is clearly equivalent to the assumption that F_{x_i} of 'f' is bounded $\forall f \in N^*$

Then by uniform boundedness theorem is $\{\|F_{x_i}\|\}$ is bounded set of numbers.

$\because$ Natural imbedding preserves norm, we have $\|F_{x_i}\| = \|x_i\|$ $\forall x_i \in X$

It follows that $\|x_i\|$ is a bounded set of numbers.

ie, $X = \{x_i\}$ is a bounded subset of N.

Hence proved.

Definition:- Let 'N' be a normed linear space and 'L' is a linear space of all scalar valued linear functions defined on 'N' and N^* is the conjugate space of N $\ni$ $N^* \subseteq L$. Let 'T' be a linear transformation from N into itself and T' is a Linear transformation of L into itself defined by $T'(f(x)) = f(T(x))$ $\forall f \in L$. Now the restriction of T' to N^* into itself is denoted by T^* and is called as "Conjugate of T" satisfies the condition that $T^*(f(x)) = f(T(x))$ where

Theorem:- If 'T' is a linear operator on a normed linear space 'N' then its conjugate T^* defined by $T^*(f(x)) = f(T(x))$ is an operator on N^* and the mapping $T \to T^*$ is an isomorphic isomorphism of $B(N) \to B(N^*)$ which reverses product & preserves an identity transformation

Proof:- Given that 'T' is a linear operator on 'N' and the conjugate space T^* defined by $T^*(f(x)) = f(T(x))$ — (1)

claim:- i, T^* is an operator on N^* $\quad$ $N^* \Rightarrow$ cont, linear transformation

ii, The mapping $T \to T^*$ is an isomorphic isomorphism of $B(N) \to B(N)$

iii, This mapping reverses product

iv, This mapping preserves an identity transformation.

To prove i:- First we prove that T^* is a linear transformation.

Let α, β be any two scalars and $f, g \in N^*$

consider $(T^*(\alpha f + \beta g))(x) = \alpha f + \beta g(T(x))$

$$= \alpha(f\, T(x)) + \beta(g\, T(x))$$

$$= \alpha T^*(f(x)) + \beta T^*(g(x))$$

$\therefore T^*(\alpha f + \beta g)(x) = \alpha T^*(f(x)) + \beta T^*(g(x))$

$\therefore T^*$ is a linear transformation.

Now we prove that T^* is continuous.

ie, we prove that T^* is bounded.

consider, $\|T^*\| = \sup\{\|T^*(f)\| : \|f\| \le 1\}$

$$\le \sup\{\|T^*(f(x))\| : \|f\| \le 1 \;\&\; \|x\| \le 1\}$$

$$= \sup\{\|f(T(x))\| : \|f\| \le 1 \;\&\; \|x\| \le 1\}$$

$$\le \sup\{\|f\|\,\|T\|\,\|x\| : \|f\| \le 1 \;\&\; \|x\| \le 1\}$$

$$\le \sup\{\|T\|$$

$\therefore \|T^*\| \leq \|T\|$ —— (3)

$\because T$ is bounded, by definition, $\exists$ a +ve integer $k \ni \|T\| \leq k \ \forall k > 0$

$\therefore (3) \Rightarrow \|T^*\| \leq k$ for $k > 0$

$\therefore T$ is bounded. And hence continuous.

$\therefore T$ is a linear operator on N^*

To prove ii:- Define a mapping $\phi: \mathbb{B}(N) \to \mathbb{B}(N^*)$ by $\phi(T) = T^*$, $\forall T \in \mathbb{B}(N)$

Now we prove that 'ϕ' is an isomorphic isomorphism.

ie, we prove that ϕ is one-one, linear transformation and $\|\phi(T)\| = \|T\|$

To prove ϕ is linear transformation:-

Let $T_1, T_2 \in \mathbb{B}(N)$ and α, β be any two scalars

Consider $\phi(\alpha T_1 + \beta T_2) = (\alpha T_1 + \beta T_2)^*$

Again consider $(\phi(\alpha T_1 + \beta T_2))^*(f(x)) = f((\alpha T_1 + \beta T_2)(x))$

$= \alpha f(T_1(x)) + \beta f(T_2(x))$

$= \alpha(T_1^* f(x)) + \beta(T_2^* f(x))$

$\therefore (\alpha T_1 + \beta T_2)^*(f(x)) = \alpha(T_1^* f(x)) + \beta(T_2^* f(x))$

$\Rightarrow (\alpha T_1 + \beta T_2)^* = \alpha T_1^* + \beta T_2^*$

$= \alpha \phi(T_1) + \beta \phi(T_2)$

$\therefore \phi$ is a linear transformation

In order to 'ϕ' is one-one. First we prove that $\|T\| = \|T^*\|$

Let 'x' be any non-zero vector in N.

Then by known theorem, for each non-zero vector in 'N' $\exists$ a functional in $N^* \ni |f(T(x))| = \|T(x)\|$ and $\|f\| = 1$

Consider $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$

$= \sup\{\|T(x)\| : \|x\| = 1\}$ where $x \neq 0$

$= \sup\left\{\dfrac{\|T(x)\|}{\|x\|} : x \neq 0\right\}$

$= \sup\left\{\dfrac{|f(T(x))|}{\|x\|} : x \neq 0, \|f\| \leq 1\right\}$

$= \sup\left\{\dfrac{|(T^*(f(x))|}{\|x\|} : x \neq 0, \|f\| \leq 1\right\}$

$\leq \sup\left\{\dfrac{\|T^*\| \|f\| \|x\|}{\|x\|} ; x \neq \{0\} \ \& \ \|f\| \leq 1\right\}$

$\therefore \|T\| \leq \|T^*\|$ —— (4)

From (3) and (4) we have $\|T\| = \|T^*\| \rightarrow (5)$

To prove ϕ is one-one:-

Consider $\phi(T_1) = \phi(T_2)$

$\Rightarrow T_1^* = T_2^*$

$\Rightarrow \|T_1^*\| = \|T_2^*\|$

$\Rightarrow \|T_1\| = \|T_2\|$ ($\because$ from (5))

$\therefore \|T_1\| - \|T_2\| = 0$

$\Rightarrow T_1 - T_2 = 0$

$\Rightarrow T_1 = T_2$

$\therefore \phi$ is one-one.

To prove $\|\phi(T)\| = \|T\|$:-

Consider $\|\phi(T)\| = \|T^*\|$ ($\because$ by (5))

$= \|T\|$

$\therefore \phi$ is an isomorphic isomorphism.

To prove iii:- ie, to prove that $\phi(T_1T_2) = \phi(T_1)\cdot\phi(T_2)$

Consider $\phi(T_1T_2) = (T_1T_2)^*$

Again consider $(T_1T_2)^* f(x) = f((T_1T_2)(x))$

$= f\,T_1(T_2(x))$

$= T_1^* f(T_2(x))$

$= T_1^* T_2^* f(x)$

$\therefore (T_1T_2)^* f(x) = T_1^* T_2^* f(x)$

$\Rightarrow (T_1T_2)^* = T_1^* T_2^*$

$\Rightarrow \phi(T_1T_2) = \phi(T_1)\,\phi(T_2)$

To prove iv:- ie, ϕ preserves an identity transformation.

Let I, be an identity operator on N.

Consider, $\phi(I) = I^*$

$I^*(f(x)) = f(I(x))$

$= f(x)$

$= I f(x)$

$\therefore I^* = I$

$\therefore \phi$ preserves an identity transformation.

Hence proved.

Section:- Hilbert's Spaces

Definition And Some Special properties:-

1, Consider the three-dimensional Eucledian space and the vector R^3 in the form $x = (x_1, x_2, x_3)$ where the norm is defined by $\|x\| = \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2}$

2, The inner product of any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in the space R^3 is defined by $(x, y) = (x_1 y_1 + x_2 y_2 + x_3 y_3)$ where the norm is defined by $(x, x) = \|x\|^2$

3, Consider the three-dimensional unitary Space C^3 then the inner product b/w any two vectors $x = (x_1, x_2, x_3)$ & $y = (y_1, y_2, y_3)$ is defined by $(x, y) = [x_1 \bar{y}_1 + x_2 \bar{y}_2 + x_3 \bar{y}_3]$ which is related to the norm $(x, x) = x_1 \bar{x}_1 + x_2 \bar{x}_2 + x_3 \bar{x}_3$

$$= |x_1|^2 + |x_2|^2 + |x_3|^2$$

$$= \|x\|^2$$

Hilbert's Space:- A Hilbert Space is a complex banach space whose norm arises from an inner product ie, in which there is defined a complex function (x, y) of vectors x and y with the following properties:-

i, $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

ii, $(\overline{x, y}) = (y, x)$

iii, $(x, x) = \|x\|^2$

Note:- From conditions i, & ii, we have $(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$

Example:- The space C_2^n with the inner product of two vectors $x = (x_1, x_2 \cdots x_n)$ $y = (y_1, y_2 \cdots y_n)$ defined by $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$ is a hilbert's space.

Sol:- given that $(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i$

In order to prove the space C_2^n is a Hilbert space. we have to prove the three conditions.

i, $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

ii, $(\overline{x, y}) = (y, x)$

iii, $(x, x) = \|x\|^2$

To prove i:- Consider $(\alpha x+\beta y, z) = \sum_{i=1}^{n} (\alpha x_i+\beta y_i) \bar{z}_i$

$= \sum_{i=1}^{n} (\alpha x_i \bar{z}_i + \beta y_i \bar{z}_i)$

$= \sum_{i=1}^{n} \alpha x_i \bar{z}_i + \sum_{i=1}^{n} \beta y_i \bar{z}_i$

$= \alpha \sum_{i=1}^{n} x_i \bar{z}_i + \beta \sum_{i=1}^{n} y_i \bar{z}_i$

$= \alpha (x,z) + \beta (y,z)$

To prove ii:- Consider $\overline{(x,y)} = \overline{\sum_{i=1}^{n} x_i \bar{y}_i}$

$= \sum_{i=1}^{n} \bar{x}_i y_i = \sum_{i=1}^{n} y_i \bar{x}_i = (y,x)$

To prove iii: Consider $(x,x) = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2$

$= \|x\|^2$

$\therefore C_2^n$ is a Hilbert's space.

Theorem:- A Schwartz Inequality:-

If x and y are any two vectors in a Hilbert space 'H' then $|(x,y)| \le \|x\| \|y\|$

Proof:- Given that x and y are any two vectors in a Hilbert space 'H'.

Claim:- $|(x,y)| \le \|x\| \|y\|$

we prove this in two cases.

Case i:- If $y=0$

$\Rightarrow \|y\| = 0$

$\Rightarrow \|x\| \|y\| = \|x\| \cdot 0 = 0$

And also $(x,y) = (x,0) = 0$

$\Rightarrow |(x,y)| = 0$

$\therefore |(x,y)| \le \|x\| \|y\|$

$\therefore$ The result is true in this case.

Case ii:- If $y \neq 0 \Rightarrow \|y\| \neq 0$

Consider, $\|x - \lambda y\|^2 \geq 0$

$\Rightarrow (x-\lambda y, x-\lambda y) \geq 0$

$\Rightarrow (x, x-\lambda y) - \lambda (y, x-\lambda y) \geq 0$

$\Rightarrow (x,x) - \bar{\lambda} (x,y) - \lambda (y,x) + \lambda \bar{\lambda} (y,y) \geq 0$

$\Rightarrow \|x\|^2 - \bar{\lambda} (x,y) - \lambda (y,x) + |\lambda|^2 \|y\|^2 \geq 0$

put $\lambda = \frac{(x,y)}{\|y\|^2}$ and $\bar{\lambda} = \frac{\overline{(x,y)}}{\|y\|^2}$

$\Rightarrow \|x\|^2 - \frac{\overline{(x,y)}}{\|y\|^2}(x,y) - \frac{(x,y)}{\|y\|^2}(y,x) + \frac{|(x,y)|^2}{\|y\|^4}\|y\|^2 \geq 0$

$\Rightarrow \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2} - \frac{(x,y)(y,x)}{\|y\|^2} + \cancel{|\lambda|^2}\frac{|(x,y)|^2}{\|y\|^2} \geq 0$

$\Rightarrow \|x\|^2 - \frac{|(x,y)|^2}{\|y\|^2} \geq 0$

$\Rightarrow \|x\|^2\|y\|^2 - |(x,y)|^2 \geq 0$

$\Rightarrow \|x\|^2\|y\|^2 \geq |(x,y)|^2$

$\Rightarrow |(x,y)|^2 \leq \|x\|^2\|y\|^2$

$\Rightarrow |(x,y)| \leq \|x\|\,\|y\|$

$\therefore$ The result is true in this case

Hence proved.

Result:- The inner product in a Hilbert space 'H' is jointly con[tinuous]

Proof:- Let us take $\{x_n\}$ and $\{y_n\}$ are two cgt sequencies.

ie, $\|x_n - x\| \to 0$ as $n \to \infty$

and $\|y_n - y\| \to 0$ as $n \to \infty$

Claim :- $(x_n, y_n) \to (x,y)$

Now consider, $|(x_n,y_n) - (x,y)| = |(x_n,y_n) - (x_n,y) + (x_n,y) - (x,y)|$

$= |(x_n, y_n - y) - (x_n - x, y)|$

$\leq |(x_n, y_n - y)| + |(x_n - x, y)|$

$\leq \|x_n\|\,\|y_n - y\| + \|x_n - x\|\,\|y\|$

$\to 0$ as $n \to \infty$

$\therefore |(x_n, y_n) - (x,y)| \to 0$

$\Rightarrow |(x_n, y_n)| \to |(x,y)|$

Hence proved.

Parallelogram law:- $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Consider $\|x+y\|^2 = (x+y, x+y)$

$= (x,x) + (x,y) + (y,x) + (y,y)$

$= \|x\|^2 + (x,y) + (y,x) + \|y\|^2$ —(1)

consider $\|x-y\|^2 = (x-y, x-y)$

$= (x,x) - (x,y) - (y,x) + (y,y)$

$= \|x\|^2 - (x-y) - (y,x) + \|y\|^2$ —(2)

(1) + (2) $\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

which is the parallelogram law

(1) − (2) $\Rightarrow \|x+y\|^2 - \|x-y\|^2 = (x,y) + (y,x) - (-(x,y) - (y,x))$

$= 2(x,y) + 2(y,x)$ ——(3)

Now replace 'y' by iy by (3)

$\|x+iy\|^2 - \|x-iy\|^2 = 2(x,iy) + 2(iy,x)$

Now multiply 'i' with on both sides

$i\|x+iy\|^2 - i\|x-iy\|^2 = 2i(x,iy) + 2i(iy,x)$

$= 2i\bar{i}(x,y) + 2ii(y,x)$

$= 2(x,y) - 2(y,x)$ ——(4)

Now (3)+(4) $\Rightarrow \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4(x,y)$

which is called polarisation identity.

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Theorem:-B A closed convex subset 'C' of a Hilbert space 'H' contains a unique vector of smallest norm.

Proof:- Given that 'C' is a closed convex subset of Hilbert's space 'H'.

claim:- 'C' contains a unique vector of smallest norm.

Since 'C' is convex.

for any $x,y \in C$ and $0 \le t \le 1$ we have $ty + (1-t)x \in C$

$\therefore C$ is non-empty.

Let us consider $t = 1/2$

$\Rightarrow \frac{x+y}{2} \in C \quad \forall x,y \in C$

Let $d = \inf\{\|x\| : x \in C\}$ ——(1)

Then $\exists$ a sequence $\{x_n\}$ in C such that $\|x_n\| \to d$ as $n \to \infty$ ——(2)

Thus, we prove that $\{x_n\}$ is a cauchy sequence.

Let $x_n, x_m \in C$

$\Rightarrow$ by definition, $\frac{x_n+x_m}{2} \in C$

by eqⁿ (1) we have $d \le \left\|\frac{x_n+x_m}{2}\right\|$

$\Rightarrow 2d \le \|x_n+x_m\|$

$\Rightarrow 4d^2 \le \|x_n+x_m\|^2$

$\Rightarrow -4d^2 \ge -\|x_n+x_m\|^2$

$\Rightarrow -\|x_n+x_m\|^2 \le -4d^2$ ——(3)

Now by applying parallelogram law, we have

$\|x_n+x_m\|^2 + \|x_n-x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2$

$\Rightarrow \|x_n-x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n+x_m\|^2$

$\Rightarrow \|x_n-x_m\|^2 \le 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2$ ($\because$ by (3))

$= 2d^2 + 2d^2 - 4d^2$ ($\because$ by ②)

$\to 0$ as $n \to \infty$

$\Rightarrow \|x_n - x_m\|^2 \to 0$ as $n \to \infty$

$\Rightarrow \|x_n - x_m\| \to 0$ as $n \to \infty$

$\therefore \{x_n\}$ is a cauchy sequence in C

Since 'C' is a closed subset of H

$\Rightarrow \{x_n\}$ is a cauchy sequence in H

Since H is complete.

$\Rightarrow x_n \to x$ in H

but 'C' is the closed subset of H

$\therefore$ C is also complete.

$\therefore x_n \to x$ in C

$\Rightarrow x = \lim_{n\to\infty} x_n$

$\Rightarrow \|x\| = \|\lim_{n\to\infty} x_n\|$

$\Rightarrow \|x\| = \lim_{n\to\infty} \|x_n\|$

$\Rightarrow \|x\| = d$

$\therefore$ 'C' contains a vector 'x' of smallest norm.

To prove uniqueness:- Suppose x' be the another vector in C with the smallest norm

ie, $\|x'\| = d$ and $x \neq x'$

Since $x, x' \in C$ and C is convex.

$\Rightarrow \frac{x+x'}{2} \in C$

by $d \leq \|\frac{x+x'}{2}\|$ — (4)

Again by applying parallelogram law, we have

$\|\frac{x}{2} + \frac{x'}{2}\|^2 + \|\frac{x}{2} - \frac{x'}{2}\|^2 = 2\|\frac{x}{2}\|^2 + 2\|\frac{x'}{2}\|^2$

$\Rightarrow \|\frac{x}{2} + \frac{x'}{2}\|^2 = 2\|\frac{x}{2}\|^2 + 2\|\frac{x'}{2}\|^2 - \|\frac{x}{2} - \frac{x'}{2}\|^2$

$\Rightarrow \|\frac{x}{2} + \frac{x'}{2}\|^2 < 2\|\frac{x}{2}\|^2 + 2\|\frac{x'}{2}\|^2 - 0$

(Since $x \neq x' \Rightarrow x - x' \neq 0 \Rightarrow \frac{x-x'}{2} > 0$

$\Rightarrow \|\frac{x-x'}{2}\|^2 > 0$

$\Rightarrow -\|\frac{x-x'}{2}\|^2 < 0$)

$\Rightarrow \|\frac{x}{2}+\frac{x'}{2}\|^2 < 2\frac{d^2}{4}+2\frac{d^2}{4}$

$= d^2$

$\Rightarrow \|\frac{x+x'}{2}\|^2 < d^2$

$\Rightarrow \|\frac{x+x'}{2}\| < d$

which is a contradiction to (4)

$\therefore x = x'$

$\therefore$ C contains a unique vector of smallest norm.

Hence proved.

Theorem:- 6 If B is a complex banach space whose norm obeys the parallelogram law and if the inner product is defined by $4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$ then prove that B is a Hilbert's space.

Proof:- Given that B is a complex banach space whose norm obeys parallelogram law

ie, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

given the inner product defined on 'B' as

$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$

claim:- B is a Hilbert's space with respect to the given inner product

ie, to prove that i, $(\alpha x+\beta y, z) = \alpha(x,z) + \beta(y,z)$

ii, $(\overline{x,y}) = (y,x)$

iii, $(x,x) = \|x\|^2$

To prove i :- To prove this we have to prove that

a, $(x+y,z) = (x,z) + (y,z)$

b, $(\alpha x, y) = \alpha(x,y)$

To prove a :-

Consider,

$4(x+y,z) = \|x+y+z\|^2 - \|x+y-z\|^2 + i\|x+y+iz\|^2 - i\|x+y-iz\|^2$ —(1)

Take $\|(x+y)+z\|^2 = \|(x+z)+y\|^2$

$= 2\|x+z\|^2 + 2\|y\|^2 - \|(x+z)-y\|^2$ —(2)

Now $\|(x+z)-y\|^2 = \|(z-y)+x\|^2$

$= 2\|z-y\|^2 + 2\|x\|^2 - \|(z-y)-x\|^2$ — (3)

Substitude eq^n (3) in (2) we have

$\|(x+y)+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - 2\|z-y\|^2 - 2\|x\|^2 + \|(z-y)-x\|^2$

$\Rightarrow \|(x+y)+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - 2\|z-y\|^2 - 2\|x\|^2 + \|(x+y)-z\|^2$

$\Rightarrow \|(x+y)+z\|^2 - \|(x+y)-z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - 2\|z+y\|^2 - 2\|x\|^2$ —(4)

Interchange 'x' and 'y'

$\Rightarrow \|(x+y)+z\|^2 - \|(x+y)-z\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - 2\|x-z\|^2 - 2\|y\|^2$ —(5)

Now (4)+(5)

$\Rightarrow 2\|(x+y)+z\|^2 - 2\|(x+y)-z\|^2 = 2\|x+z\|^2 - 2\|x-z\|^2 + 2\|y+z\|^2 - 2\|y-z\|^2$

$\Rightarrow 2[\|(x+y)+z\|^2 - \|(x+y)-z\|^2] = 2[\|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2]$

$\Rightarrow \|(x+y)+z\|^2 - \|(x+y)-z\|^2 = \|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2$ —(6)

Replace 'z' by iz and multiply with 'i' in eq^n (6)

$i\|(x+y)+iz\|^2 - i\|(x+y)-iz\|^2 = i\|x+iz\|^2 - i\|x-iz\|^2 + i\|y+iz\|^2 - i\|y-iz\|^2$ —(7)

(6) + (7) $\Rightarrow$

$\|(x+y)+z\|^2 - \|(x+y)-z\|^2 + i\|(x+y)+iz\|^2 - i\|(x+y)-iz\|^2 = \|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2 + i\|x+iz\|^2 - i\|x-iz\|^2 + i\|y+iz\|^2 - i\|y-iz\|^2$

$\Rightarrow 4(x+y, z) = 4(x,z) + 4(y,z)$

$\therefore (x+y, z) = (x,z) + (y,z)$

To prove b:- ie, $(\alpha x, y) = \alpha(x,y)$

Depending on 'α' we have four cases.

I. α is a +ve integer:-

we prove this by induction.

The case I, is trivial for $n=1$

Now we prove the result is true for $n=2$

we know that $(x+y, z) = (x,z) + (y,z)$

put $y = x$

$\Rightarrow (x+x, z) = (x,z) + (x,z)$

$\Rightarrow (2x, z) = 2(x,z)$

Now assume the result is true for 'n'

ie, $(nx, y) = n(x,y)$

Now we prove the result for $n+1$

consider. $((n+1)x, y) = (nx+x, y)$

$$= (nx, y) + (x, y)$$

$$= n(x, y) + (x, y)$$

$$= (n+1)(x, y)$$

$\therefore$ The result is true for $n+1$

$\therefore$ The result is true $\forall$ +ve integers.

ii. If 'α' is -ve integer:-

Consider,

$4(-x, y) = \|-x+y\|^2 - \|-x-y\|^2 + i\|-x+iy\|^2 - i\|-x-iy\|^2$

$\Rightarrow 4(-x, y) = \|x-y\|^2 - \|x+y\|^2 + i\|x-iy\|^2 - i\|x+iy\|^2$

$\Rightarrow 4(-x, y) = -[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2]$

$\Rightarrow 4(-x, y) = -4(x, y)$

$\Rightarrow (-x, y) = -(x, y)$ —— (8)

Take $\alpha = -\beta$, β is a +ve integer

consider,

$(\alpha x, y) = (-\beta x, y)$

$= -(\beta x, y)$ $\quad (\because$ by (8))

$= -\beta(x, y)$ $\quad (\because$ by case i.)

$= \alpha(x, y)$ $\quad (\because -\beta = \alpha)$

$\therefore (\alpha x, y) = \alpha(x, y)$

iii, If 'α' is a rational number:-

ie, $\alpha = \frac{p}{q}$ where p, q are integers and $q \neq 0$

consider, $(\alpha x, y) = (\frac{p}{q}x, y)$

$= (pz, y)$ where $z = \frac{x}{q}$

$(\alpha x, y) = p(z, y)$ as 'p' is an integer —(9)

Also $(x, y) = (qz, y)$ $(\because z = \frac{x}{q} \Rightarrow x = qz)$

$(x, y) = q(z, y)$ as 'q' is an integer.

$\Rightarrow (z, y) = \frac{1}{q}(x, y)$ — (10)

Substitude (10) in (9) we get

$(\alpha x, y) = p\frac{1}{q}(x, y)$

$= \frac{p}{q}(x, y)$

$\therefore (\alpha x, y) = \alpha(x, y)$ $\quad (\because \frac{p}{q} = \alpha)$

iv, If α is a complex number:-

Such that $\alpha = \alpha_1 + i\alpha_2$ where α_1, α_2 are reals.

consider,

$$4(ix, y) = \|ix+y\|^2 - \|ix-y\|^2 + i\|ix+iy\|^2 - i\|ix-iy\|^2$$

$$= |i|^2\|x+iy\|^2 - |i|^2\|x-iy\|^2 + i|i|^2\|x+y\|^2 - i|i|^2\|x-y\|^2$$

$$= i\|x+y\|^2 - i\|x-y\|^2 - \|x+iy\|^2 + \|x-iy\|^2 \quad (\because |i|^2 = 1)$$

$$= i\left[\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right]$$

$$= i\,4(x,y)$$

$\Rightarrow 4(ix, y) = i4(x,y)$

$\Rightarrow (ix, y) = i(x,y)$

Now consider,

$$(\alpha x, y) = ((\alpha_1 + i\alpha_2)x, y)$$

$$= (\alpha_1 x + i\alpha_2 x, y)$$

$$= (\alpha_1 x, y) + (i\alpha_2 x, y)$$

$$= \alpha_1(x,y) + i\alpha_2(x,y)$$

$$= (\alpha_1 + i\alpha_2)(x,y)$$

$$= \alpha(x,y)$$

$\therefore (\alpha x, y) = \alpha(x,y)$

To prove ii:-

ie, prove that $\overline{(x,y)} = (y,x)$

$$4\overline{(x,y)} = \overline{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2}$$

$$= \overline{\|x+y\|^2} - \overline{\|x-y\|^2} + \overline{i\|x+iy\|^2} - \overline{i\|x-iy\|^2}$$

$$= \|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2$$

$$= \|y+x\|^2 - \|y-x\|^2 - i|i|^2\|y+ix\|^2 + i|i|^2\|y-ix\|^2$$

$$= \|y+x\|^2 - \|y-x\|^2 + i\|y+ix\|^2 - i\|y-ix\|^2$$

$$= 4(y,x)$$

$\therefore 4\overline{(x,y)} = 4(y,x)$

$\therefore \overline{(x,y)} = (y,x)$

To prove iii:- ie, prove that $(x,x) = \|x\|^2$

consider, $4(x,x) = \|x+x\|^2 - \|x-x\|^2 + i\|x+ix\|^2 - i\|x-ix\|^2$

$= \|2x\|^2 - 0 + i\|x(1+i)\|^2 - i\|x(1-i)\|^2$

$= 4\|x\|^2 + i|1+i|^2\|x\|^2 - i|1-i|^2\|x\|^2$

$= 4\|x\|^2 + i\,2\|x\|^2 - i\,2\|x\|^2$

$= 4\|x\|^2$

$\therefore 4(x,x) = 4\|x\|^2$

$\therefore (x,x) = \|x\|^2$

$\therefore$ B is a Hilbert's space.

Hence proved.

Orthogonal Compliments

Definition:- Let 'x' and 'y' are two vectors in a Hilbert's space 'H' are said to be orthogonal if $(x,y) = 0$

Note:- If 'x' is orthogonal to y. we can represent them $x \perp y$ where the symbol $\perp$ is pronounced as $\perp$lar.

Properties:-

1. If $x \perp y$ then $y \perp x$

Since $x \perp y$

by definition, $(x,y) = 0$

$\Rightarrow \overline{(x,y)} = 0$

$\Rightarrow (y,x) = 0$

$y \perp x$ (by definition)

2. $x \perp 0$ for each x in H

Since $(x,0) = 0$

by definition, $x \perp 0$

3. If $x \perp x \Rightarrow x = 0$

Since $x \perp x$

by definition, $(x,x) = 0$

$\Rightarrow \|x\|^2 = 0$

$\Rightarrow \|x\| = 0$

$\Rightarrow x = 0$

4) Pythagoras Theorem:- If $x \perp y$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2 = \|x-y\|^2$

Proof:- consider, $\|x+y\|^2 = (x+y, x+y)$

$= (x, x+y) + (y, x+y)$

$= (x,x) + (x,y) + (y,x) + (y,y)$

$= \|x\|^2 + 0 + 0 + \|y\|^2$

$= \|x\|^2 + \|y\|^2$

$\therefore \|x+y\|^2 = \|x\|^2 + \|y\|^2$

Again consider,

$\|x-y\|^2 = (x-y, x-y)$

$= (x, x-y) - (y, x-y)$

$= (x,x) - (x,y) - (y,x) + (y,y)$

$= \|x\|^2 - 0 - 0 + \|y\|^2$

$= \|x\|^2 + \|y\|^2$

$\therefore \|x-y\|^2 = \|x\|^2 + \|y\|^2$

$\therefore \|x+y\|^2 = \|x\|^2 + \|y\|^2 = \|x-y\|^2$

Hence proved.

Definition:- A vector 'x' is said to be orthogonal to a non empty set 'S'. If $(x,y)=0$ for every $y \in S$ and we write it as $x \perp S$

Definition:- The orthogonal compliment of 'S' is the set of all vectors orthogonal to 'S' and is denoted by $S^{\perp}$ i.e, $S^{\perp} = \{x : x \perp S\}$

Results:-

1) $\{0\}^{\perp} = H$

Always we have $\{0\}^{\perp} \subseteq H$ — (1)

Let $x \in H$

Since $x \perp 0 \ \forall x \in H$

$\Rightarrow (x,0) = 0$

$\Rightarrow x \in \{0\}^{\perp}$ ($\because$ by definition)

$\therefore H \subseteq \{0\}^{\perp}$

From (1) and (2) we have

$H = \{0\}^{\perp}$

3. $H^{\perp} = \{0\}$

Proof:- Let $x \in H^{\perp}$

by definition, $x \perp H$

$\Rightarrow (x, y) = 0 \ \forall \ y \in H$

Take $y = x$

$\Rightarrow (x, x) = 0$

$\Rightarrow \|x\|^2 = 0$

$\Rightarrow \|x\| = 0$

$\Rightarrow x = 0$

$\therefore H^{\perp} = \{0\}$

3. Let 'S' and '$S^\perp$' are two closed linear subspaces of Hilbert space 'H' then $S \cap S^\perp \subseteq \{0\}$

Proof:- Let $x \in S \cap S^\perp$

$\Rightarrow x \in S$ and $x \in S^\perp$

By definition $x \perp S$

ie, x is orthogonal to itself

$\Rightarrow (x, x) = 0$

$\Rightarrow \|x\|^2 = 0$

$\Rightarrow x = 0$

$\therefore x \in \{0\} \Rightarrow S \cap S^\perp \subseteq \{0\}$

4. Let 'S' and '$S^\perp$' are two closed linear subspaces of a Hilbert space 'H' if $0 \in S$ and $0 \in S^\perp$ then $S \cap S^\perp = \{0\}$

Proof:- By above result, we have $S \cap S^\perp \subseteq \{0\}$ — (1)

$\because 0 \in S$ and $0 \in S^\perp$

$\Rightarrow \{0\} \subseteq S \cap S^\perp$ — (2)

from (1) & (2) $S \cap S^\perp = \{0\}$

5. Let S_1, S_2 be two non-empty subsets of a Hilbert space 'H' if $S_1 \subseteq S_2$ then $S_2^\perp \subseteq S_1^\perp$

Proof:- Let $x \in S_2^\perp$

$\Rightarrow$ By definition, $x \perp S_2$

$\Rightarrow x$ is $\perp$'ar to every vector in S_2

$\Rightarrow x$ is $\perp$'ar to every vector in S_1

$\Rightarrow x \in S_1^\perp$

$\therefore S_2^\perp \subseteq S_1^\perp$

6. Let 'S' be a non-empty subset of a Hilbert space 'H' then $S^\perp$ ~~orthogonal to~~ is a closed linear subspace of H.

Proof:- Given that 'S' is a non-empty subset of a Hilbert space 'H'

Claim:- $S^\perp$ is a closed linear subspace of H

Since $S^\perp = \{x / x \perp S\}$

ie, $(x, y) = 0 \ \forall \ y \in S$

Since $0 \in H$ and $(0, y) = 0 \ \forall \ y \in S$

$\Rightarrow 0 \in S^\perp$

$\therefore S^{\perp}$ is non-empty.

Now we prove that $S^{\perp}$ is a linear sub-space.

Let $x_1, x_2 \in S^{\perp}$

$$\left.\begin{array}{l}\Rightarrow \text{By definition, } (x_1, y) = 0 \ \forall\, y \in S \\ \Rightarrow \text{By definition, } (x_2, y) = 0 \ \forall\, y \in S\end{array}\right\} \longrightarrow (1)$$

Let α, β be any scalars.

Consider $(\alpha x_1 + \beta x_2, y) = (\alpha x_1, y) + (\beta x_2, y)$

$$= \alpha (x_1, y) + \beta (x_2, y)$$

$$= \alpha (0) + \beta (0)$$

$$= 0 \Rightarrow (\alpha x_1 + \beta x_2, y) = 0 \ \forall\, y \in S$$

$\therefore (\alpha x_1 + \beta x_2) y) \in S^{\perp}$

$\therefore S^{\perp}$ is a linear subspace.

To prove $S^{\perp}$ is closed :- Let 'x' be a limit point of $S^{\perp}$

we prove that $x \in S^{\perp}$

ie, $(x, y) = 0 \ \forall\, y \in S$

$\because x$ is a limit point of $S^{\perp}$

$\exists$ a sequence $\{x_n\}$ in $S^{\perp}$ $\ni x_n \to x$ as $n \to \infty$

ie, $\lim_{n\to\infty} x_n = x$

Since $x_n \in S^{\perp}$ we have $(x_n, y) = 0 \ \forall\, y \in S$

Now consider $(x, y) = (\lim_{n\to\infty} x_n, y)$

$$= \lim_{n\to\infty} (x_n, y)$$

$$= \lim_{n\to\infty} 0 \ \forall\, y \in S$$

$$= 0$$

$\therefore (x, y) = 0 \ \forall\, y \in S$

$\therefore x \in S^{\perp}$

$\therefore S^{\perp}$ is closed.

Hence proved.

4, $S \subseteq (S^{\perp})^{\perp} = S^{\perp\perp}$

Proof:- Since $S^{\perp} = \{x \mid x \perp S\}$

ie, $(x, y) = 0 \ \forall\, y \in S$, and $S^{\perp\perp} = \{x \mid x \perp S^{\perp}\}$

ie, $(x, y) = 0 \ \forall\, y \in S^{\perp}$

Let $x \in S$

$\Rightarrow (x, y) = 0 \ \forall\, y \in S^{\perp}$

$\Rightarrow x \in S^{\perp\perp}$

$\therefore S \subseteq S^{\perp\perp}$

8. Let 'M' be a closed linear subspace of a Hilbert space,
then $M^{\perp}$ is also a closed linear subspace of H and $M \cap M^{\perp} = \{0\}$

Proof:- By result-6 we have $M^{\perp}$ is a closed linear subspace of 'H'.

To prove $M \cap M^{\perp} = \{0\}$

Since M and $M^{\perp}$ are closed linear subspaces of H, we have
$0 \in M$ and $0 \in M^{\perp}$

$\Rightarrow \{0\} \subseteq M \cap M^{\perp}$ —— (1)

Let $x \in M \cap M^{\perp}$

$\Rightarrow x \in M$ and $x \in M^{\perp}$

$\Rightarrow (x, x) = 0$

$\Rightarrow \|x\|^2 = 0$

$\Rightarrow \|x\| = 0$

$\Rightarrow x = 0$

$\therefore M \cap M^{\perp} \subseteq \{0\}$ —(2)

$\therefore$ from (1) and (2) $M \cap M^{\perp} = \{0\}$

Theorem:- A Let 'M' be a closed linear subspace of a Hilbert Space 'H'. Let 'x' be a vector not in M and 'd' be the distance from x to M then $\exists$ a unique vector y_0 in M $\ni \|x - y_0\| = d$

Proof:- given that 'M' be a closed linear subspace of a Hilbert Space 'H'.

Also given that 'x' be a vector not in M and 'd' be the distance from x to M.

claim:- $\exists$ a unique vector y_0 in M $\ni \|x - y_0\| = d$

Let $C = \{x + m : m \in M\} = x + M$

then 'C' is a closed convex set, for C is convex

Let $x + m_1, x + m_2 \in C$, $0 \le t \le 1$

consider, $(1-t)(x+m_1) + t(x+m_2) = x + m_1 - t\cancel{x} - tm_1 + t\cancel{x} + tm_2$

$= x + m_1 - tm_1 + tm_2$

$= x + m_1(1-t) + tm_2$

$= x + (m_1(1-t) + tm_2) \in C$

$\therefore$ 'C' is convex.

for 'C' is closed.

Since M is closed $\Rightarrow x+M$ is also closed.

$\Rightarrow C$ is closed.

Then by known theorem, (Theorem-B) $\exists$ a unique vector $z_0 \in C$ with the smallest norm $\ni \|z_0\| = d$

Let $y_0 = x - z_0$

$z_0 = x - y_0 \in C = x+M$

$\Rightarrow x - y_0 \in x+M$

$\Rightarrow -y_0 \in M$

$\Rightarrow -(-y_0) \in M$

$\Rightarrow y_0 \in M$

and $\|z_0\| = \|x-y_0\| = d$

To prove Uniqueness:- Let y_1 be the another vector in $M \ni \|x-y_1\| = d$ and $y_1 \neq y_0$

Then corresponding to y_1 in M $\exists$ another vector z_1 in $C \ni z_1 = x - y_1$ where $z_0 \neq z_1$ & $\|z_1\| = \|x-y_1\| = d$

which is a contradiction to the uniqueness of z_0 with the smallest norm.

$\therefore y_0$ is the unique vector in $M \ni \|x-y_0\| = d$

Hence proved.

Theorem-B: If 'M' is a proper closed linear subspace of a Hilbert space 'H' then $\exists$ a non-zero vector z_0 in $H \ni z_0 \perp M$

Proof:- Given that 'M' is a proper closed linear subspace of a Hilbert space 'H'.

Claim:- $\exists$ a non-zero vector z_0 in $H \ni z_0 \perp M$.

Let 'x' be the vector not in M and 'd' be the distance from x to M.

Then by above theorem, $\exists$ a unique vector y_0 in $M \ni \|x-y_0\| = d$

Now define $z_0 = x - y_0 \in H$

$\Rightarrow \|z_0\| = \|x-y_0\| = d > 0$

$\therefore z_0$ is a non-zero vector in H.

Now we prove that $z_0 \perp M$

ie, we prove that if 'y' be any arbitrary vector in M then $z_0 \perp y$

Since $y_0 \in M$ and $y \in M$

$\Rightarrow \alpha y \in M$ for any scalar 'α'

$\Rightarrow y_0 + \alpha y \in M$

By definition of 'd' we have, $\|x - (y_0 + \alpha y)\| \geq d = \|z_0\|$

$$\Rightarrow \|x - y_0 - \alpha y\| \geq \|z_0\|$$

$$\Rightarrow \|z_0 - \alpha y\| \geq \|\overset{z_0}{\cancel{x-y_0}}\|$$

$$\Rightarrow \|z_0\|^2 \leq \|z_0 - \alpha y\|^2$$

$$\Rightarrow (z_0 - \alpha y, z_0 - \alpha y) \geq \|z_0\|^2$$

$$\Rightarrow (z_0, z_0) - (z_0, \alpha y) - (\alpha y, z_0) + (\alpha y, \alpha y) \geq \|z_0\|^2$$

$$\Rightarrow \cancel{\|z_0\|^2} - \bar{\alpha}(z_0, y) - \bar{\alpha}(y, z_0) + \|\alpha y\|^2 \geq \cancel{\|z_0\|^2}$$

$$\Rightarrow -\bar{\alpha}(z_0, y) - \bar{\alpha}(\overline{z_0, y}) + |\alpha|^2\|y\|^2 \geq 0$$

Take $\alpha = \beta(z_0, y)$ where 'β' is arbitrary.

$$\Rightarrow -\overline{\beta(z_0, y)}^{(z_0,y)} - \beta(z_0, y)(\overline{z_0, y}) + |\alpha|^2\|y\|^2 \geq 0$$

$$\Rightarrow -\overline{\beta(z_0, y)}(z_0, y) - \overline{\beta(z_0, y)}(\overline{z_0, y}) + |\beta|^2|(z_0, y)|^2\|y\|^2 \geq 0$$

$$\Rightarrow -\beta|(z_0, y)|^2 - \beta|(z_0, y)|^2 + |\beta|^2|(z_0, y)|^2\|y\|^2 \geq 0$$

$$\Rightarrow -2\beta|(z_0, y)|^2 + |\beta|^2\|y\|^2|(z_0, y)|^2 \geq 0$$

Let $|(z_0, y)|^2 = a$ and $\|y\|^2 = b$

$\Rightarrow -2\beta a + |\beta|^2 b a \geq 0$

$\Rightarrow a\beta(-2 + b|\beta|) \geq 0$ — (1)

if $a > 0$, then the left hand side of eq^n (1) becomes zero for sufficiently for positive 'β'

which is a contradiction.

and hence $a = 0 \Rightarrow |(z_0, y)|^2 = 0$

$$\Rightarrow |(z_0, y)| = 0$$

$$\Rightarrow (z_0, y) = 0$$

$\Rightarrow z_0 \perp y$, y is arbitrary element in M.

$\therefore z_0 \perp M$

Hence proved.

Theorem:- C If 'M' and 'N' are closed linear subspaces of a Hilbert space 'H' such that $M \perp N$ then the linear space $M+N$ is also closed.

Proof: Given that 'M' and 'N' are closed linear subspaces of a Hilbert space 'H' $\ni M \perp N$

claim:- The linear subspace $M+N$ is closed.

Clearly, $M+N$ is a subspace of H

In order to prove $M+N$ is closed.

we have to prove that $M+N$ contains all of its limit points

Let 'z' be a limit point of $M+N$

we have to prove $z \in M+N$

since 'z' is a limit point of $M+N$

$\Rightarrow \exists$ a sequence $\{z_n\}$ in $M+N$ $\ni z_n$ converges to z as $n \to \infty$

Since $z_n \in M+N$

$\Rightarrow$ each z_n in $M+N$ can be written in this form.

$z_n = x_n + y_n$ where $x_n \in M$, $y_n \in N$

consider $\|z_n - z_m\|^2 = \|(x_n+y_n)-(x_m+y_m)\|^2$

$$= \|(x_n - x_m) + (y_n - y_m)\|^2$$

$$\leq \|x_n - x_m\|^2 + \|y_n - y_m\|^2$$

Since 'x_n' and 'y_n' are cauchy sequences in M and N respectively

$\Rightarrow \{x_n\} \to x$, $\{y_n\} \to y$

$\therefore z_n$ is a cauchy-sequence

Since M and N are closed and hence they are complete.

ie, M and N contains all of its limit points.

$\therefore \exists$ a vector x in M and y in N $\ni x_n \to x$ and $y_n \to y$

$\Rightarrow x_n + y_n \to x+y$

where $x+y \in M+N$

Since $z = \lim_{n\to\infty} z_n$

$= \lim_{n\to\infty} (x_n + y_n)$

$= \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$

$= x+y \in M+N$

$\Rightarrow z \in M+N$ $\therefore M+N$ is closed.

Hence Proved.

Theorem:- D If M is closed linear subspace of H then $H = M \oplus M^{\perp}$

Proof:- Given that M is a closed linear subspace of H

claim:- $H = M \oplus M^{\perp}$

By result - 8 we have $M^{\perp}$ is also closed linear subspace of H

and $M \cap M^{\perp}$ is empty.

Since M and $M^{\perp}$ are two closed linear subspaces of H.

Then by theorem - C $M + M^{\perp}$ is also closed linear subspace of H.

Now we have to prove that $H = M \oplus M^{\perp}$

Since $M \cap M^{\perp} = \{0\}$

$\Rightarrow M$ and $M^{\perp}$ are linearly independent

Then it remains to prove that $H = M + M^{\perp}$

If it is not possible then $M + M^{\perp}$ is a proper subspace of H

ie, $M + M^{\perp} \subseteq H$

Then by known theorem-(B) $\exists$ a non zero vector z_0 in $H \ni$

$z_0 \perp M + M^{\perp} \in M^{\perp} + M^{\perp\perp}$

$\Rightarrow z_0 \in M^{\perp}$ and $z_0 \in M^{\perp\perp}$

$\Rightarrow z_0 \in M^{\perp} \cap M^{\perp\perp}$

But which is contradiction to the fact that $M^{\perp} \cap M^{\perp\perp}$ is empty

$M^{\perp} \cap M^{\perp\perp} = \{0\}$ and $z_0 \neq 0$

$\therefore H = M \oplus M^{\perp}$

Hence proved.

* Orthonormal Sets *

Definition:- An orthonormal set in a Hilbert space 'H' is a non empty subset of H which consists of mutually orthogonal unit vectors.

ie, it is a non-empty subset $\{e_i\}$ of 'H' if the following properties

i, $\forall i \neq j$ $e_i \perp e_j$

ii, $\|e_i\| = 1$ $\forall i$

Note:- 1, If 'H' contains only the zero vector (ie, $H = \{0\}$) then 'H' has no orthonormal sets.

2, If 'H' contains a non-zero vector 'x' and if we normalize 'x' by considering $e = \frac{x}{\|x\|}$ then the single element set $\{e\}$

is an orthonormal set

3 In general if $\{x_i\}$ is a non-empty set of mutually orthogonal non-zero vectors in 'H' and each x_i is normalised as $e_i = \frac{x_i}{\|x_i\|}$. Then the resulting set e_i is an orthonormal set

Theorem:- A Bessel's Inequality:-

Let $\{e_1, e_2, e_3 ---- e_n\}$ be a finite orthonormal set in a Hilbert space 'H'. If 'x' is any vector in H then $\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$, furthermore $x - \sum_{i=1}^{n} (x,e_i) e_i \perp e_j \ \forall j$

Proof:- Given that $S = \{e_1, e_2, e_3 --- e_n\}$ be a finite orthonormal set in a Hilbert space 'H'.

And 'x' is any vector in H.

claim:- 1, $\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$, Furthermore

2, $x - \sum_{i=1}^{n} (x,e_i) e_i \perp e_j \ \forall j$

To prove 1:- case i:- If 'x' is a zero vector in H

ie, $x = 0$

$\Rightarrow \|x\| = 0$

$\Rightarrow \|x\|^2 = 0$

and $(x, e_i) = 0$

$\Rightarrow |(x,e_i)|^2 = 0$

$\therefore$ The result is true in this case.

case ii:- If x is a non-zero vector in H

consider $\|x - \sum_{i=1}^{n} |(x,e_i)| e_i\|^2 \geq 0$ ($\because$ norm is non-ve)

$\Rightarrow (x - \sum_{i=1}^{n} (x,e_i) e_i, x - \sum_{j=1}^{n} (x,e_j) e_j) \geq 0$ for $i=j$

$\Rightarrow (x, x) - \sum_{i=1}^{n} (x,e_i)(e_i,x) - \sum_{j=1}^{n} (x,e_j)(\overline{x,e_j}) + \sum_{i=1}^{n}\sum_{j=1}^{n} (x,e_j)(\overline{x,e_j})(e_i,e_j) \geq 0$

$\Rightarrow \|x\|^2 - \sum_{i=1}^{n} (x,e_i)(\overline{x,e_i}) - \sum_{j=1}^{n} |(x,e_j)|^2 + \sum_{j=1}^{n} (x,e_j)(\overline{x,e_j})(e_i,e_j) \geq 0$ for $i=j$

$\Rightarrow \|x\|^2 - \sum_{i=1}^{n} |(x,e_i)|^2 - \sum_{j=1}^{n} |(x,e_j)|^2 + \sum_{j=1}^{n} |(x,e_j)|^2 \geq 0$ [$\because \|(e_i,e_j)\|^2 = 1$]

$\|x\|^2 - \sum_{i=1}^{n} |(x,e_i)|^2 \geq 0$

$\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$

To prove 2:- Consider $(x - \sum_{i=1}^{n} (x,e_i) e_i, e_j) = (x,e_j) - \sum_{i=1}^{n} (x,e_i)(e_i,e_j)$

$= (x,e_j) - \sum_{i=1}^{n} (x,e_i)(e_i,e_j)$

$= (x, e_j) - [(x, e_1)(e_1, e_j) + (x, e_2)(e_2, e_j) + \cdots + (x, e_j)(e_j, e_j) \cdots + (x, e_n)(e_n, e_j)]$

$= (x, e_j) - (x, e_j)(1)$ $\left[\because \begin{array}{ll} (e_i, e_j) = 0 & \forall i \neq j \\ (e_i, e_j) = 1 & \forall i = j \end{array}\right]$

$= (x, e_j) - (x, e_j)$

$= 0$

$\therefore x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j \ \forall j$

Hence proved.

Theorem:- B If $\{e_i\}$ is an orthonormal set in a Hilbert Space 'H' and if 'x' is any vector in 'H' then the sets $\{e_i \mid (x, e_i) \neq 0\}$ is empty or countable.

Proof:- Given that $\{e_i\}$ is an orthonormal set in a Hilbert space 'H' and x is any vector in H.

claim:- $S = \{e_i \mid (x, e_i) \neq 0\}$ is empty or countable

for each +ve integer 'n', $S_n = \{e_i \mid |(x, e_i)|^2 > \frac{\|x\|^2}{n}\}$

By bessel's inequality, we have S_n has atmost $(n-1)$ vectors

For, if S_n has 'n' vectors say $e_1, e_2, e_3 \cdots e_n$

Now $|(x, e_1)|^2 > \frac{\|x\|^2}{n}$

$|(x, e_2)|^2 > \frac{\|x\|^2}{n}$

$\cdots\cdots$

$\cdots\cdots$

$|(x, e_n)|^2 > \frac{\|x\|^2}{n}$

$\Rightarrow \sum_{i=1}^{n} |(x, e_i)|^2 > \left(\frac{\|x\|^2}{\cancel{n}}\right)\cancel{n}$

$\Rightarrow \sum_{i=1}^{n} |(x, e_i)|^2 > \|x\|^2$

which is a contradiction to the Bessel's inequality.

Now consider $S = \bigcup_{n=1}^{\infty} S_n$

Thus, we have 'S' is either empty or countable.

Theorem:- C General form of Bessel's Inequality:-

Statement:- If $\{e_i\}$ is an orthonormal set in a Hilbert Space 'H' then $\sum |(x, e_i)|^2 \leq \|x\|^2$ for every vector x in 'H'.

Proof:- Given that $\{e_i\}$ is an orthonormal set in a Hilbert space 'H'.

For every vector x in H we have to prove that $\Sigma |(x,e_i)|^2 \leq \|x\|^2$ — (1)

If 'x' is a zero vector then the result is trivally true.

If 'x' is a non-zero vector in H.

Let us define $S = \{e_i \mid (x,e_i) \neq 0\}$

Then by Theorem-B we have 'S' is empty or countable.

If 'S' is empty:-

ie, $\exists$ no sequence $\{e_i\} \ni (x,e_i) \neq 0$

$\Rightarrow (x,e_i) = 0 \ \forall\ i$

Since x is a non-zero vector we have $\|x\| > 0$

$\therefore \Sigma |(x,e_i)|^2 \leq \|x\|^2$

If 'S' is countable:- Then we have again two cases.

ie, either 'S' is finite or countably infinite.

If 'S' is finite:- Then the set 'S' can be written as

$S = \{e_1, e_2, e_3 \cdots e_n\}$ for some +ve integer 'n'.

In this case we define $\sum_{i \in u} |(x,e_i)|^2 = \sum_{i=1}^{n} |(x,e_i)|^2$

which is clearly independent of the order in which the elements of 'S' are arranged.

$\therefore$ The inequality eq^n (1) can be written as $\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$

which is proved by theorem-A

If 'S' is countably infinite:- Let the vectors in 'S' can be arranged in definite order.

ie, $S = \{e_1, e_2, e_3 \cdots e_n\}$

Then by the theory of absolutely convergent series if $\sum_{n=1}^{\infty} |(x,e_n)|^2$ cgs then every series obtained by re-arranging its terms is also cgs. And all such series have same sum.

Now define $\Sigma |(x,e_i)|^2 = \sum_{i=1}^{\infty} |(x,e_i)|^2$

$\Rightarrow \sum_{S} |(x,e_i)|^2$ is a non-negative extended real number which depends only on 'S' but not on the arrangement of the vectors in 'S'.

Now the proof is complete if we show that $\sum_{n=1}^{\infty} |(x,e_n)|^2 \leq \|x\|^2$

Since from case I; No partial sum of the series on the left hand side of eqⁿ ① can exceed $\|x\|^2$

$\Rightarrow$ it is clearly follows that eqⁿ ① is true.

i.e., $\sum_{n=1}^{\infty} |(x,e_n)|^2 \leq \|x\|^2$

Hence proved.

Theorem:- D If $\{e_i\}$ is an orthonormal set in a Hilbert space and if 'x' is any vector in 'H' then $x - \sum((x,e_i))e_i$ is orthogonal to e_j for each 'j'

Proof:- Given that $\{e_i\}$ is an orthonormal set in a Hilbert space 'H' and x is any vector in H

Claim:- $x - \sum(x,e_i)e_i \perp e_j$ for each 'j' —— (1)

Consider the set $S = \{e_i \mid (x,e_i) \neq 0\}$

Again By Theorem-B S is either empty or countable.

If 'S' is empty:-

$\Rightarrow (x,e_i) = 0 \ \forall i$

$\Rightarrow (x,e_i)e_i = 0 \ \forall i$

$\Rightarrow \sum(x,e_i)e_i = 0 \ \forall i$

$\Rightarrow (x-0)$ is orthogonal to e_j (by eqⁿ ①)

$\Rightarrow x \perp e_j$

which is true because, 'S' is empty.

Hence the theorem is true in this case.

If 'S' is countable:-

Then either 'S' is finite or countabally infinite.

If 'S' is finite:- Then 'S' can be written as $S = \{e_1, e_2, \ldots e_n\}$

~~Let the~~ Now define $\sum(x,e_i)e_i = \sum_{i=1}^{n}(x,e_i)e_i$

-proof in Exam

Then by condition (ii) in Theorem-A our claim is trivally true

If 'S' is countabally infinite:-

Then the vectors in 'S' can be arranged in definite order i.e, $S = \{e_1, e_2, e_3 \ldots e_n \ldots\}$

Let us take $S_n = \sum_{i=1}^{n} (x, e_i) e_i$ then for some $m > n$

consider, $\|-S_n + S_m\|^2 = \|-\sum_{i=1}^{n} (x, e_i) e_i + \sum_{i=1}^{m} (x, e_i) e_i\|^2$

$= \|\sum_{i=n+1}^{m} (x, e_i) e_i\|^2$

$= (\sum_{i=n+1}^{m} (x, e_i) e_i, \sum_{i=n+1}^{m} (x, e_i) e_i)$

$= \sum_{i=n+1}^{m} |(x, e_i)|^2$

$\leq \|x\|^2$ ($\because$ By Bessel's inequality)

$\Rightarrow \|S_m - S_n\|^2 \leq \|x\|^2$

Since the series $\sum_{n=1}^{\infty} |(x, e_n)|^2$ cgs we have the series $\sum_{i=1}^{m} |(x, e_i)|^2$ is also converges.

$\Rightarrow \|S_m - S_n\|^2 \to 0$ as $m, n \to \infty$

$\Rightarrow \|S_m - S_n\| \to 0$ as $m, n \to \infty$

$\therefore \{S_n\}$ is a cauchy sequence in H and since 'H' is complete $\{S_n\}$ cgs to a limit point S in H where $S = \sum_{n=1}^{\infty} (x, e_n) e_n$

Since $S_n \to S$ we have $\lim_{n \to \infty} S_n = S$

Now define $\sum (x, e_i) e_i = \sum_{n=1}^{\infty} (x, e_n) e_n$ then our claim is true

By cond. ii in theorem-(A)

consider, $(x - \sum (x, e_i) e_i, e_j) = (x - \sum_{n=1}^{\infty} (x, e_n) e_n, e_j) = (x - S), e_j$

$= (x, e_j) - (S, e_j)$

$= (x, e_j) - (\lim_{n \to \infty} S_n, e_j)$

$= (x, e_j) - \lim_{n \to \infty} (S_n, e_j)$

$= (x, e_j) - \lim_{n \to \infty} (\sum_{i=1}^{n} (x, e_i) e_i, e_j)$

$= (x, e_j) - \lim_{n \to \infty} (\sum_{i=1}^{\infty} (x, e_i) e_i, e_j)$

$= (x, e_j) - \sum_{i=1}^{\infty} (x, e_i) (e_i, e_j)$

$= (x, e_j) - (x, e_j)$

$= 0$

$\therefore (x - \sum (x, e_i)) e_i \perp e_j \ \forall j$

Now the proof is complete if we show that this definition $\sum (x, e_i) e_i$ is valid.

In the sense that, it does not depend on the arrangement of the vectors in 'S'.

Let the vectors in 'S' be arranged in a manner that $S_1 = \{f_1, f_2, f_3 \ldots f_n\}$ where $S_n' = \sum_{i=1}^{n} (x, f_i) f_i$

And we see the sequence $\{S_n'\} \to S'$ where $S' = \sum_{n=1}^{\infty} (x, f_n) f_n$

Now we conclude the proof by showing $S' = S$

Let $\epsilon > 0$ be given

Since $S_n \to S$ and $S_n' \to S'$

By definition, $\|S_n - S\| < \epsilon/3$ —(2) and $\|S_n' - S'\| < \epsilon/3$ —(3) for $n \geq n_0$

where n_0 is a +ve integer.

For some +ve $m_0 > n_0$ all terms of S_{n_0} occurs among these S_{m_0}'

So $S_{m_0}' - S_{n_0}$ is a finite sum of terms of the form $(x, e_i) e_i$

for $i = n_0+1, n_0+2, \ldots m_0$

i.e., $S_{m_0} - S_{n_0} = \sum_{i=n_0+1}^{m_0} (x, e_i) e_i$

consider $\|S_{m_0} - S_{n_0}\|^2 = \|\sum_{i=n_0+1}^{m_0} (x, e_i) e_i\|^2$

$$= \left(\sum_{i=n_0+1}^{m_0} (x, e_i) e_i, \sum_{i=n_0+1}^{m_0} (x, e_i) e_i\right)$$

$$= \sum_{i=n_0+1}^{m_0} \|(x, e_i)\|^2 (e_i, e_i)$$

$$= \sum_{i=n_0+1}^{m_0} \|(x, e_i)\|^2$$

$$< (\epsilon/3)^2$$

[Since the series $\sum_{i=1}^{\infty} |(x, e_i)|^2$ converges

$\Rightarrow \sum_{i=n_0+1}^{m_0} |(x, e_i)|^2$ also converges.

$\Rightarrow \sum_{i=n_0+1}^{m_0} |(x, e_i)|^2 < (\epsilon/3)^2$ —(4)]

$\Rightarrow \|(S_{m_0} - S_{n_0})\|^2 < (\epsilon/3)^2$

$\Rightarrow \|S_{m_0} - S_{n_0}\| < \epsilon/3$ —(5)

Now consider $\|S' - S\| = \|S' - S_{m_0}' + S_{m_0} + S_{n_0} - S_{n_0} - S\|$

$$\leq \|S' - S_{m_0}'\| + |S_{m_0}' - S_{n_0}\| + \|S_{n_0} - S\|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3$$

$$< \epsilon$$

$\therefore \|S' - S| < \epsilon$ as ϵ - arbitrary $S' = S$

Hence proved.

Definition:- An orthonormal set $\{e_i\}$ in 'H' is said to be complete if it is maximal ie, if it is impossible to adjoint a vector $e \rightarrow \{e_i\}$ in such a way that $\{e, e_i\}$ is an orthonormal set which properly contains $\{e_i\}$

Theorem-E :- Every non-zero hilbert space 'H' contains a complete orthonormal set

Proof:- Let $S = \{\frac{x}{\|x\|}\}$ is an orthonormal set in 'H'.

Let 'p' be the set of all orthonormal sets containing 'S'. Then 'p' is non-empty because atleast 'S' lies in p.

Now 'Q' be the class of all orthonormal sets.

Then clearly, Q is a partially ordered set which respect to set inclusion and p has a upper bound in Q.

$\therefore$ p satisfies all the conditions of zorn's lemma.

By zorn's lemma $\exists$ must exists a maximal element. Let it be 'M'.

$\therefore$ M is a complete orthonormal set containing 'S'.

For, if not so. Then $\exists$ another orthonormal set containing 'S' & also M properly.

This will contradict the maximal character of 'M'.

$\therefore$ H contains a complete orthonormal set.

Theorem:- F Let 'H' be a hilbert space and let $\{e_i\}$ be an orthonormal set in H. Then the following conditions are equivalent to one another.

i, $\{e_i\}$ is complete

ii, $x \perp \{e_i\} \Rightarrow x = 0$

iii, if x is an arbitrary vector in H then $x = \Sigma (x, e_i) e_i$

iv, if x is an arbitrary vector in H then $\|x\|^2 = \Sigma |(x, e_i)|^2$

Proof:- Given that 'H' be a hilbert space.

Let $\{e_i\}$ be an orthonormal set in 'H'.

Claim:- Given four conditions are equivalent

To prove i, $\Rightarrow$ ii, :-

Assume i, ie, $\{e_i\}$ is complete.

Claim:- $x \perp \{e_i\} \Rightarrow x = 0$

Since $\{e_i\}$ is complete $\Rightarrow e_i$ is maximal in H

ie, we can adjoint a vector e to e_i in such a way that (e, e_i) is an orthonormal set —(1)

Suppose that i, is not true.

ie, $x \perp \{e_i\} \Rightarrow x \neq 0$

Since $x \perp \{e_i\} \Rightarrow (x, e_i) = 0$

Now define a vector 'e' by $e = \frac{x}{\|x\|}$

Then (e, e_i) is an orthonormal set.

for i, consider, $(e, e_i) = \left(\frac{x}{\|x\|}, e_i\right)$

$= \frac{1}{\|x\|}(x, e_i)$

$\therefore (e, e_i) = 0$

$\Rightarrow e \perp e_i$

ii, $\|e\| = \left\|\frac{x}{\|x\|}\right\| = \frac{\|x\|}{\|x\|} = 1$

But which is a contradiction to eqn (1)

$\therefore x \perp \{e_i\} \Rightarrow x = 0$

To prove ii $\Rightarrow$ iii:-

Assume ii, ie, $x \perp \{e_i\} \Rightarrow x = 0$

Claim:- $x = \Sigma(x, e_i)e_i \ \forall \ x$ in H

By theorem-B we have $x - \Sigma(x, e_i)e_i \perp e_j \ \forall j$, for every vector x in H)

$\Rightarrow x - \Sigma(x, e_i)e_i = 0$ ($\because$ by supposition)

$\Rightarrow x = \Sigma(x, e_i)e_i$

To prove iii $\Rightarrow$ iv:-

Assume iii ie, $x = \Sigma(x, e_i)e_i \ \forall \ x$ in H

Consider $\|x\|^2 = \|\Sigma(x, e_i)e_i\|^2 = (x, x)$

$= (\Sigma(x, e_i)e_i, \Sigma(x, e_i)e_i)$

$= \Sigma|(x, e_i)|^2 (e_i, e_i)$

$= \Sigma|(x, e_i)|^2$

$\therefore \|x\|^2 = \Sigma|(x, e_i)|^2$

To prove iv $\Rightarrow$ i:-

Assume iv, ie, $\|x\|^2 = \Sigma|(x, e_i)|^2 \ \forall \ x$ in H

we have to prove that $\{e_i\}$ is complete.

On the contrary, suppose that $\{e_i\}$ is not complete.

$\Rightarrow (e, e_i)$ is an orthonormal set which properly contains e_i

Since (e, e_i) is an orthonormal set

By definition, $e \perp e_i$ and $\|e\| = 1$

Since $e \perp e_i \Rightarrow (e, e_i) = 0$ $(\because x = \Sigma(x, e_i))$ replace x by e

Consider $\|e\|^2 = \Sigma|(e, e_i)|^2$

$= 0$

$\Rightarrow \|e\|^2 = 0$

$\Rightarrow \|e\| = 0$

which is a contradiction to $\|e\| = 1$

$\therefore \{e_i\}$ is complete.

Hence proved.